NONLINEAR ELECTROMAGNETIC PHENOMENA IN A LIQUID WITH NON-EQUILIBRIUM ELECTRICAL CONDUCTIVITY IN AN ALTERNATING MAGNETIC FIELD

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On the basis of [1] this note examines nonlinear electromagnetic phenomena in a dense plasma brought about by the variation in its electrical conductivity as the electrical field changes.

It is well known that the electrical conductivity depends on the electric field strength due to the following causes. The electrons in moving in the electric field receive energy from the field which may be considerable over the free path length. However it is difficult for this energy to be transferred to the heavy particles. In monatomic gases the energy exchange between electrons and heavy particles comes about basically as a result of elastic collisions. Thus a noticeable difference in electron and ion temperature, determined by the electron energy balance taking radiation losses into account, turns out to be possible even for relatively weak electric fields. In molecular gases, on the other hand, the fundamental energy exchange mechanism is the excitation of the rotational and oscillatory degrees of freedom of the molecules. Thus the electron energy in these gases is dissipated relatively easily, and the electron temperature is not observed to be noticeably higher than the atomic temperature.

The concept of the characteristic "plasma field" E_p is introduced in [2], which is determined for an isotropic plasma by the relation

$$E_p = \sqrt{3kTme^{-2}\delta(\omega^2+\nu_0^2)}\,.$$

Here k is the Boltzmann constant, T is the plasma temperature in the absence of a field, m and e are the electronic charge and mass, δ is the mean fraction of energy transferred to a heavy particle by an electron on collision, ω is the frequency of field variation, ν_0 is the electron-ion collision frequency in the absence of a field.

In weak electromagnetic fields ($E \ll E_p$) the plasma maintains thermodynamic equilibrium, and the electrical conductivity of the plasma is independent of the field. In strong electric fields ($E \gg E_p$) there is a sharp difference of electron temperature and the voltage-current characteristics of the plasma become nonlinear.

The question of nonequilibrium electrical conductivity has been fairly fully studied [3–5] as regards monatomic gas plasmas like argon and potassium mixtures. It was shown in [3] that for the plasmas which were considered the dependence of the electrical conductivity on the electric field with no magnetic field present could be satisfactorily described by a power function of the absolute current density, i.e., $\sigma = c\,|\mathbf{j}|^{\gamma}$, where c is a function of the atomic temperature. This function has also been confirmed experimentally for an argon-potassium plasma for a temperature of the order of 0.2 eV and a pressure of the order of 1 atm. [3].

In the following we consider electromagnetic phenomena in a dense plasma with an electrical conductivity of the type $\sigma = c\,|\mathbf{j}|^{\gamma}$ when it is in motion in a traveling magnetic field. It is assumed that the plasma parameters and limits of variation of the independent quantities (j, T_e) are such that the function $\sigma = c\,|\mathbf{j}|^{\gamma}$ is stable [4]. In addition the plasma is taken as having the properties of an ideal incompressible fluid. These last assumptions together with the assumption that the gradients of static pressure and pondermotive forces are only in the direction of plasma motion allow us to commence from the equations of electrodynamics.

1. We shall write the electromagnetic field equations in the form of the induction equation

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = \mu_0\sigma\left(-\frac{\partial\mathbf{A}}{\partial t} + \mathbf{u}\times\operatorname{rot}\mathbf{A}\right), \qquad (1.1)$$

$$\sigma = c\left|-\frac{\partial\mathbf{A}}{\partial t} + \mathbf{u}\times\operatorname{rot}\mathbf{A}\right|^{\frac{\gamma}{1-\gamma}},$$

(cont'd)

$$\operatorname{rot}\mathbf{A} = \mathbf{B}, \qquad \operatorname{div}\mathbf{A} = 0 \qquad (0<\gamma<1). \qquad (1.1)$$

We introduce the dimensionless quantities

$$\mathbf{A}^\times = \frac{\mathbf{A}}{A_0}, \quad \sigma^\times = \frac{\sigma}{\sigma_0}, \quad \mathbf{u}^\times = \frac{\mathbf{u}}{u_0}, \quad t^\times = 2\pi f_0 t,$$

$$\operatorname{rot}^\times = \frac{\lambda_0}{2\pi}\operatorname{rot} \qquad \varepsilon_0 = \frac{\mu_0\sigma_0 f_0}{2\pi}\lambda_0^2. \qquad (1.2)$$

Here λ_0 is the length of a characteristic wave, u_0 is the phase velocity of this wave for a characteristic frequency f_0,

$$\sigma_0 = cE_0^{\frac{\gamma}{1-\gamma}}, \qquad E_0 = 2\pi f_0 A_0. \qquad (1.3)$$

Equation (1.1) takes the form

$$-\operatorname{rot}^\times\operatorname{rot}^\times\mathbf{A}^\times =$$

$$= \varepsilon_0\left|\frac{\partial\mathbf{A}^\times}{\partial t^\times} - \mathbf{u}^\times\times\operatorname{rot}^\times\mathbf{A}^\times\right|^{n-1}\left(\frac{\partial\mathbf{A}^\times}{\partial t^\times} - \mathbf{u}^\times\times\operatorname{rot}^\times\mathbf{A}^\times\right),$$

$$n = \frac{1}{1-\gamma}. \qquad (1.4)$$

We have in mind configurations having cylindrical symmetry. For such configurations we obtain from (1.4)

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A) =$$

$$= \varepsilon_0\left|\frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x}\right|^{n-1}\left(\frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x}\right). \qquad (1.5)$$

It is assumed here that due to symmetry the vector-potential has an azimuthal component only, the upper index $\times$ being discarded.

In connection with problems considered below it is assumed that the boundary conditions have the form

$$A|_{\rho_1} = \varphi_1(x-t), \qquad A|_{\rho_2} = \varphi_2(x-t),$$

where, in a particular case, (φ_1, φ_2) may be understood as traveling waves of the type sin (x − t). We shall thus look for solutions of Eq. (1.5) in a class of functions in which the variables x and t appears only in the combination $\tau = x - t$. We may then write (1.5) as

$$\frac{\partial^2 A}{\partial\tau^2} + \frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A) = \delta\varepsilon_0 s^n\left(\frac{\partial A}{\partial\tau}\right)^n,$$

$$s = u - 1, \qquad \delta = \left(\operatorname{sign}\frac{\partial A}{\partial\tau}\right)^{n-1}(\operatorname{sign} s)^{n-1}. \quad (1.6)$$

Proceeding according to the general method of obtaining invariant-group solutions (H-solutions), we shall find the H-solutions of Eq. (1.6) in one-parameter subgroups.

2. We replace Eq. (1.6) by the equivalent system

$$\frac{\partial u^1}{\partial x^1}+\frac{\partial u^2}{\partial x^2}-\delta\varepsilon_0 s^n\left(\frac{\partial u^3}{\partial x^1}\right)^n=0, \qquad \frac{\partial u^3}{\partial x^1}-u^1=0,$$
$$\frac{1}{x^2}\frac{\partial (x^2u^3)}{\partial x^2}-u^2=0, \qquad x^1=\tau,\ x^2=\rho,\ u^3=A. \quad (2.1)$$

Calculating the infinitesimal operators in a one-parameter group we have

$$X=\xi_x{}^i\frac{\partial}{\partial x^i}+\xi_u{}^k\frac{\partial}{\partial u^k} \qquad (i=1,2;\ k=1,2,3). \quad (2.2)$$

Solving the defining equations gives the following values for the coordinates of allowable operators:

$$\xi_x{}^1=(1-n)c_0x^1+c_1, \qquad \xi_u{}^1=c_0u^1,$$
$$\xi_u{}^3=(2-n)c_0u^3+c_2x^2+\frac{c_3}{x^2},$$
$$\xi_x{}^2=(1-n)c_0x^2, \qquad \xi_u{}^2=c_0u^2\,2c_2,\ . \quad (2.3)$$

Here c_i are arbitrary constants.

Thus in accordance with (2.3) the fundamental group of system (2.1) is generated by the following linearly independent infinitesimal operators:

$$X_1=(1-n)x^1\frac{\partial}{\partial x^1}+(1-n)x^2\frac{\partial}{\partial x^2}+$$
$$+u^1\frac{\partial}{\partial u^1}+u^2\frac{\partial}{\partial u^2}+(2-n)u^3\frac{\partial}{\partial u^3},$$
$$X_2=\frac{\partial}{\partial x^1}, \quad X_3=2\frac{\partial}{\partial u^2}+x^2\frac{\partial}{\partial u^3}, \quad X_4=\frac{1}{x^2}\frac{\partial}{\partial u^3}. \quad (2.4)$$

In order to find all possible fundamentally different solutions of rank 1, we must find the optimum system of first order subgroups. The operators corresponding to this system and simultaneously satisfying the condition for the existence of H-solutions are

$$X_1,\ X_2,\ X_2+\alpha X_3,\ X_2+\alpha X_4. \quad (2.5)$$

Here and in what follows α and α_1 are arbitrary parameters.

In addition to the operators (2.5) we shall, for considerations which will become clear later on, consider a somewhat different solution of rank 1 based on the subgroup

$$X_2+\alpha X_3+\alpha_1X_4. \quad (2.6)$$

The H-solutions for the separate subgroups are given below with the exception of one for which an H-solution could not be obtained. These solutions are given for the variable $u^3 = A$ (the variables u^1, u^2 play an auxiliary part). The subgroup generated by the operator X is denoted by H[X].

1. H[X_1]. The H-solution for this subgroup satisfies the equation

$$(1+\lambda^2)\frac{d^2\omega}{d\lambda^2}-\frac{n-3}{n-1}\lambda\frac{d\omega}{d\lambda}-\frac{2n-3}{(1-n)^2}\omega-\delta\varepsilon_0 s^n\left(\frac{d\omega}{d\lambda}\right)^n=0,$$
$$\lambda=\frac{x^1}{x^2}, \qquad u^3=(x^2)^{\frac{n-2}{n-1}}\omega. \quad (2.7)$$

2. H[X_2].

$$u^3=\gamma_1|x^2|+\gamma_2|x^2|^{-1}. \quad (2.8)$$

Here and elsewhere γ_1 and γ_2 are arbitrary constants.

3. H[$X_2 + \alpha X_3$].

$$u^3=\delta\varepsilon_0(s\alpha)^n\frac{(x^2)^{2+n}}{(1+n)(3+n)}+\alpha x^1x^2+\frac{\gamma_1}{x^2}+\gamma_2x^2. \quad (2.9)$$

4. H[$X_2 + \alpha X_4$].

$$u^3=\delta\varepsilon_0(s\alpha)^n\frac{(x^2)^{2-n}}{(1-n)(3-n)}+\alpha\frac{x^1}{x^2}+\frac{\gamma_1}{x^2}+\gamma_2x^2. \quad (2.10)$$

5. H[$X_2 + \alpha X_3 + \alpha_1X_4$].

$$u^3=\frac{M}{x^2}f(x^2)+\left(x^2+\frac{a}{x^2}\right)\alpha x^1+\frac{\gamma_1}{x^2}+\gamma_2\left(x^2+\frac{a}{x^2}\right),$$
$$f(x^2)=\int\frac{[(x^2)^2+a]^n}{(x^2)^{n-2}}dx^2-(x^2)^2\int\frac{[(x^2)^2+a]^n}{(x^2)^n}dx^2,$$
$$a=\frac{\alpha_1}{\alpha}, \qquad M=-\frac{\delta\varepsilon_0(s\alpha)^n}{2}. \quad (2.11)$$

3. We shall consider the applicability of the H-solutions which have been obtained to boundary problems.

1. The plasma moves along an infinitely long cylindrical tube of radius ρ_2 in the direction of the x axis, the vector-potential on the tube has a φ component.

2. The plasma occupies a space the internal boundary of which is a cylinder of radius ρ_1, the plasma (or cylinder) moves in the direction of the x axis and the vector potential on the cylinder has a φ component.

3. The plasma moves in the direction of the x axis along a cylindrical channel of infinite length formed by two coaxial cylinders of radii ρ_1 and ρ_2. The vector potential on both cylinders has a φ component.

In all these cases it is assumed that the boundary conditions represent a traveling wave. The vector-potential in the plasma has to be found.

It can be easily seen that none of the H-solutions which have been obtained are suitable as an exact solution of these problems. However an approximate solution may still be obtained, and the H-solutions (2.9)–(2.11) may be employed for this purpose.

The H-solutions considered above are valid for any interval Δx^1, and so the boundary conditions over separate intervals may be changed to segments of the straight line $\beta_i x^1$ and the parameters α_i and α_{1_i} used (up till now they have remained arbitrary) for matching the H-solutions with the boundary conditions. The intervals in this procedure should be small. We thus obtain a simple algorithm for the calculation.

The boundary conditions for H-solution (2.9) should be written in the form

$$u^3|_{\rho_1=0}=0, \qquad u^3|_{\rho_2}=\beta_2x^1.$$

From here

$$u^3=\frac{\delta\varepsilon_0(s\beta_2)^n}{(1+n)(3+n)}\left[\frac{(x^2)^{2+n}}{\rho_2{}^n}-\rho_2x^2\right]+\frac{\beta_2}{\rho_2}x^1x^2. \quad (3.1)$$

Consequently (3.1) gives a solution of the first problem.

The boundary conditions for the H-solution (2.10) should have the form

$$u^3|_{\rho_1}=\beta_1x^1, \qquad u^3|_{\rho_2=\infty}=0 \quad \text{(radiation condition)}.$$

From here it follows that for n > 2

$$u^3=\frac{\delta\varepsilon_0(s\beta_1)^n}{(1-n)(3-n)}\left[(x^2)^{2-n}\rho_1{}^n-\frac{\rho_1{}^3}{x^2}\right]+\beta_1\rho_1\frac{x^1}{x^2}. \quad (3.2)$$

Consequently (3.2) gives the solution of the second problem.

The boundary conditions for the H-solution (2.11) are

$$u^3|_{\rho_1}=\beta_1x^1, \qquad u^3|_{\rho_2}=\beta_2x^1.$$

Therefore

$$u^3 = \frac{M}{x^2(\rho_1^2 - \rho_2^2)} \{\rho_1^2 [f(x^2) - f(\rho_2)] - \rho_2^2 [f(x^2) - f(\rho_1)] + (x^2)^2 [f(\rho_2) - f(\rho_1)]\} + \left(\alpha x^2 + \frac{\alpha_1}{x^2}\right) x^1,$$

$$\alpha = \frac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1^2 - \rho_2^2}, \qquad \alpha_1 = \frac{\rho_1\beta_2 - \rho_2\beta_1}{\rho_1^2 - \rho_2^2}\rho_1\rho_2. \qquad (3.3)$$

We thus have the solution of the third problem.

Remembering that (3.1)–(3.3) give solutions for any of the intervals, the parameters β_1 and β_2 should be determined from the boundary conditions in their initial form for each Δx^1. For boundary conditions of the traveling wave type the solutions given in (3.1)–(3.3) are periodic in the variable x^1; for even n these solutions have finite discontinuities.

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